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Programme 2
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'1/0' IS NOT NONSENSE - A NOVEL WAY TO INTERPRET TERMS, SUCH THAT EVERY TERM IS DEFINED

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Rapport de recherche

Programme 2 (Calcul symbolique, programmation et genie logiciel)

Projet Eureca (Environnement integre de specification et de preuve)

Titre (en francais)

`` 1 / 0 '' n'est pas absurde ---
Une nouvelle facon d'interpreter les termes,
de sorte que chaque terme soit defini

Resume (en francais)

Les termes, tels qu'ils sont ecrits et lus par les informaticiens, les algebristes et les logiciens, sont des noms composes, c'est-a-dire des sequences grammaticalement correctes de noms propres et de variables. Bien qu'il soit facile d'ecrire et de copier un terme, aussi facile que le terme est court, beaucoup de termes paraissent si etranges, par exemple `` 1 / 0 '', que la plupart des gens les considerent absurdes. Pourtant, de tels termes apparaissent partout dans les programmes informatiques, et dans leurs specifications. Il y a donc au moins une bonne raison pour uniformiser le domaine, et pour adopter une procedure d'intepretation qu'on puisse executer dans tous les cas sans echec, c'est-a-dire faire en sorte qu'on soit capable de comprendre tous les termes sans exception.

‘ 1/0 ’ *is not nonsense*

—

A novel way to interpret terms, such that every term is defined

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Summary

Terms, as written and read by computer programmers, algebraists or logicians, are composite names, i.e., syntactically correct sequences of proper names and variables. Although all terms are very easy to write and copy, as easy as the term is short, many seem so strange, the monster ‘ 1/0 ’ being at the extreme, that most people consider them nonsense. Yet, they appear everywhere in specifications of computer programs. So there is at least one good reason to smooth the field down, and adopt an interpretation-procedure executing which you never fail, i.e., you always understand the term, any term.

Keywords

Denotation, ill-defined, logic, model theory, nonsense, partial algebra, partial function, semantics, semi-function, term, undefined.

Structure of this text

1 PRELUDE

α — Most procedures are *partial*, for example these: (each procedure is represented by a term)

```
x / y
factorial ( x )
x ^ y
x-array [ y ]
pop ( x-stack )
```

which you, or the computer, cannot execute if you choose these arguments:

```
1 / 0
factorial ( -1 )
0 ^ 0
array [ 1000000000 ]
pop ( empty )
```

So, a lot of ground terms do not correspond to any computation.

The situation is even worse than that. There are *composite procedures* that are not executable *whatever the arguments.* For example these:

```
factorial ( -1 - x ^ 2 )
pop pop push ( x , empty )
```

I.e., a lot of terms correspond to *non-executable* procedures; a lot of programs that one could come up with are not executable.

β — In front of that, what do people do? A common and reasonable idea is to ban those terms, those programs, as ill-formed or nonsense. But how do you distinguish the bad ones? Unless you have seen them before and remember, *only too late,* i.e., after you have started computing.

Therefore I advocate another idea, less common (section *An incomplete but significant survey of comparable ideas*; see below) but I believe more fruitful: not only accept to read any term, but also prepare to *reason* about it; not only accept that there are buggy programs, there are indeed, but be able to understand systematically and precisely how buggy that one is.

γ — How to implement the idea? Up to now you have considered two sorts of things: (1) *texts,* either terms or programs; and (2) *procedures,* i.e., classes of computations (actions, transformations). You need now to consider another sort of ingredients: (3) *mathematical objects,* i.e., ideal eternal things.

Here is the whole intention:

> model every *procedure* in *mathematics* (the model of a procedure being a mathematical quasi-function), then switch over to interpreting every *text* as not a procedure but its mathematical model.

The first half of that project is delicate and difficult. — The second half, to interpret every term as a semi-function, is our subject...

2 IMAGINING SEMI-FUNCTIONS, CONTEMPLATING TERMS

2.1 Elementary concepts

2.1.1 Functional graph

Every set of pairs, not two pairs sharing the same first component, is called '*functional graph*' [4]. I.e., every functional graph G has these two qualities:
(1) there is a set U_1, there is a set U_2, $G \subseteq U_1 \times U_2$ (G is a set of pairs);
(2) $\forall (x_1, y_1) \in G$, $\forall (x_2, y_2) \in G$,

$$x_1 = x_2 \quad \Rightarrow \quad y_1 = y_2$$

Examples
1 — $\emptyset$ is a functional graph.
2 — You know the three real numbers 0, 1 and $e = \exp 1$. This set:

$$\{\, ((0,1),0),\ ((1,1),1),\ ((e,1),e),\ ((0,e),0),\ ((1,e),1/e),\ ((e,e),1)\,\}$$

is a functional graph (think of the division restricted to $\{0,1,e\}$). □

Notation (the domain of a graph) — $\operatorname{Dom} G = \{x \in U_1 \mid \exists (x,y) \in G\}$. Verify immediately that $\operatorname{Dom} \emptyset = \emptyset$.

Example

$$
\begin{aligned}
& \operatorname{Dom}\{((0,1),0),\ ((1,1),1),\ ((e,1),e),\ ((0,e),0),\ ((1,e),1/e),\ ((e,e),1)\} \\
= \ & \{(0,1),\ (1,1),\ (e,1),\ (0,e),\ (1,e),\ (e,e)\}
\end{aligned}
$$

□

2.1.2 Semi-function (partial function)

What I call *'semi-function '* is what most people call 'partial function' [13]. I do not like the phrase 'partial function' because, although every function is a semi-function, *most semi-functions are not functions* (think of 'group' and 'semi-group', for example).

Definition 1 (semi-function) — *I call every triplet,* (G, E, F)*,*
- G *being a functional graph,*
- E *being a set,* F *being a set,*
- $G \subseteq E \times F$*,*

'a semi-function'.

Observe easily that $\operatorname{Dom} G \subseteq E$.

Example — This triplet:

$$
(\ \{((0,1),0),\ ((1,1),1),\ ((e,1),e),\ ((0,e),0),\ ((1,e),1/e),\ ((e,e),1)\},\ \{0,1,e\}^2,\ \mathbf{R}\ )
$$

is a semi-function. □

Notation

Let f be a semi-function. By definition, f is some triplet (G, E, F).
1 — To mention E and F, I write:

$$f : E \rightharpoonup F$$

(read 'the base of f is E, its target F'; notice that I do not draw an arrow but a *harpoon)*.
2 — $\operatorname{Dom} f = \operatorname{Dom} G$; i.e., the domain of a semi-function is the domain of its graph. A semi-function whose domain is $\emptyset$, I call it 'an *empty* semi-function'.

2.1.3 Function

Every semi-function, (G, E, F), which has this quality:

$$\operatorname{Dom} G = E$$

is called *'function '* [4]. To be more informative, I write in this case:

$$f : E \rightarrow F$$

(I draw an arrow) instead of '$f : E \rightharpoonup F$'.

2.1.4 Semi-nunction

Traditionally, logicians do not interpret a constant as an element *but as a nullary function* [23]: then, of course, they liken the nullary function to the element. So let me generalize straight: interpret a constant, i.e., a semi-function-symbol whose arity is 0, as *a nullary semi-function.*
Abbreviation — *Semi-nunction* = nullary semi-function.

Notation
1 — *The 0-tuple* — Choose once and for all some object, for example $\emptyset$, or the star Arcturus as I. That element, I note it '$\diamond$' (Shoenfield [23] notes it '$()$' and calls it 'the 0-tuple').
2 — *F to the power 0* — Let F be any set; $F^0 = \{\diamond\}$.

Definition 2 (semi-nunction) — *Let F be a set. I call every semi-function $F^0 \rightharpoonup F$* 'a F-semi-nunction'.

Examples
1 — $(\emptyset, \mathbf{Q}^0, \mathbf{Q})$ is an empty $\mathbf{Q}$-semi-nunction.
2 — $(\{(\diamond, e)\}, \mathbf{R}^0, \mathbf{R})$ is an $\mathbf{R}$-semi-nunction, which looks very much like the number e. □

Semi-nunction is a useful concept, because you are now able to interpret *every* symbol, even if it does not correspond to any element of some set: for example, referring to the real numbers, interpret the symbol '`pi`' as the non-empty semi-nunction $(\{(\diamond, \pi)\}, \mathbf{R}^0, \mathbf{R})$; referring to the naturals, interpret it as the empty semi-nunction $(\emptyset, \mathbf{N}^0, \mathbf{N})$. That essential possibility is connected to this...

Let F be a set; let c be an F-semi-nunction.
Either $\mathrm{Dom}\, c = \emptyset$, i.e., $c = (\emptyset, F^0, F)$.
Or $\mathrm{Dom}\, c \neq \emptyset$; i.e., $c = (\{(\diamond, k)\}, F^0, F)$, k being an element of F. — I note that element '$c\diamond$'.

Observe that there is: only one *empty* F-semi-nunction; and as many *non-empty* F-semi-nunctions as there are elements of F.

2.2 Considering semi-functions

2.2.1 Vocabulary

Definition 3 (vocabulary) — *I call every pair, $(\mathcal{F}, a)$,*
- *$\mathcal{F}$ being a set of symbols (the semi-function-symbols),*
- *a being a function $\mathcal{F} \to \mathrm{N}$ (arity),*

'a vocabulary'.

Notation — Let n be a natural number. $\mathcal{F}_n = \{\mathrm{f} \in \mathcal{F} \mid a(\mathrm{f}) = n\}$ (the n-ary semi-function-symbols).

Example — Vocabulary ${}_1\mathcal{V}$

Consider this vocabulary, ${}_1\mathcal{V}$:
- $\mathcal{F}_0 = \{\mathbf{zer}, \mathbf{one}, \mathbf{exo}\}$;
- $\mathcal{F}_2 = \{\mathsf{/}, \mathsf{joy}\}$. □

2.2.2 Magma (partial algebra)

Let $\mathbb{f} \in \mathcal{F}_n$. You are now ready to choose $\mathbf{f}$ as the name of an n-ary semi-function.

Definition 4 (magma) — *Let $\mathcal{V}$ be a vocabulary; $\mathcal{V} = (\mathcal{F}, a)$. I call every pair, (Ω, Φ),*
- *Ω being a set (the universe),*
- *Φ being an indexed family of semi-functions, $\Phi = \{\phi_{\mathbf{f}}\}_{\mathbf{f} \in \mathcal{F}}$,*

which has these qualities:
(i) *$\Omega \neq \emptyset$ (the universe is not empty);*
(ii) *$\forall \mathbf{f} \in \mathcal{F}$, $\phi_{\mathbf{f}}$ is a semi-function $\Omega^{a(\mathbf{f})} \rightharpoonup \Omega$ (the arity of the semi-function is the arity of the symbol);*
'a $\mathcal{V}$-magma'.

Examples — $\mathcal{V}$-magmas ${}_1\boldsymbol{\Omega}$ and ${}_2\boldsymbol{\Omega}$

You know the vocabulary ${}_1\mathcal{V}$ (see above). Here are two ${}_1\mathcal{V}$-magmas...

1 — ${}_1\boldsymbol{\Omega} = ({}_1\Omega, {}_1\Phi)$
- ${}_1\Omega = \mathbf{R}$ (the real numbers).
- $\phi_{\mathbf{zer}}$ is this non-empty $\mathbf{R}$-semi-nunction: $\phi_{\mathbf{zer}}\diamond = 0$.
- $\phi_{\mathbf{one}}$ is this non-empty $\mathbf{R}$-semi-nunction: $\phi_{\mathbf{one}}\diamond = 1$.
- $\phi_{\mathbf{exo}}$ is this non-empty $\mathbf{R}$-semi-nunction: $\phi_{\mathbf{exo}}\diamond = e$.
- $\phi_{\mathbf{/}}$ is this semi-function (division):

$$\begin{array}{rcl} \mathbf{R}^2 & \rightharpoonup & \mathbf{R} \\ \mathbf{R} \times \mathbf{R}^* \ni (\xi, \eta) & \mapsto & \xi/\eta \end{array}$$

('$\mathbf{R}^*$' denotes the set of non-zero reals).
- $\phi_{\mathbf{joy}}$ is this semi-function:

$$\begin{array}{rcl} \mathbf{R}^2 & \rightharpoonup & \mathbf{R} \\ \mathbf{R} \times \mathbf{R}_+^* \ni (\xi, \eta) & \mapsto & \xi + \ln \eta \end{array}$$

('$\mathbf{R}_+^*$' denotes the set of positive reals).

2 — ${}_2\boldsymbol{\Omega} = ({}_2\Omega, {}_2\Phi)$
- ${}_2\Omega = \mathbf{Q}$ (the rational numbers).
- $\phi_{\mathbf{zer}}$ is this non-empty $\mathbf{Q}$-semi-nunction: $\phi_{\mathbf{zer}}\diamond = 0$.
- $\phi_{\mathbf{one}}$ is this non-empty $\mathbf{Q}$-semi-nunction: $\phi_{\mathbf{one}}\diamond = 1$.
- $\phi_{\mathbf{exo}}$ is *the empty* $\mathbf{Q}$-semi-nunction.
- $\phi_{\mathbf{/}}$ is this semi-function (division):

$$\begin{array}{rcl} \mathbf{Q}^2 & \rightharpoonup & \mathbf{Q} \\ \mathbf{Q} \times \mathbf{Q}^* \ni (\xi, \eta) & \mapsto & \xi/\eta \end{array}$$

- $\phi_{\mathbf{joy}}$ is this semi-function:

$$\begin{array}{rcl} \mathbf{Q}^2 & \rightharpoonup & \mathbf{Q} \\ \mathbf{Q} \times \mathbf{Q}^* \ni (\xi, \eta) & \mapsto & \xi - (1/\eta) \end{array}$$

□

2.3 Considering composites of semi-functions

You have just considered the magma ${}_1\Omega$. In particular you have considered the three *numbers* 0, 1 and e; and the two *semi-functions* $\phi_{/}$ and $\phi_{\mathtt{joy}}$. *How can you designate them?* You can designate each of them by its name, i.e., write either '`zer`', '`one`', '`exo`', '`/`' or '`joy`'.

Without referring to anything else, you can consider other numbers as well, for example these:

$$\begin{array}{lcl} 2 & = & 1+\ln e = \phi_{\mathtt{joy}}(1,e) \\ 3 & = & (1+\ln e)+\ln e = \phi_{\mathtt{joy}}(\phi_{\mathtt{joy}}(1,e),e) \\ \ln 2 & = & 0+\ln(1+\ln e) = \phi_{\mathtt{joy}}(0,\phi_{\mathtt{joy}}(1,e)) \end{array}$$

and consider other semi-functions, for example this one:

$$\begin{array}{ccl} \mathbf{R}^3 & \rightharpoonup & \mathbf{R} \\ (\xi,\eta,\zeta) & \mapsto & \xi+\ln(\eta+\ln\zeta) = \phi_{\mathtt{joy}}(\xi,\phi_{\mathtt{joy}}(\eta,\zeta)) \end{array}$$

How can you designate those other objects? *You can designate them by terms!*

`joy one exo`	(the number 2)
`joy ( joy one exo ) exo`	(the number 3)
`joy zer ( joy one exo )`	(the number $\ln 2$)
`joy x ( joy y z )`	(the semi-function; see above)

(I have chosen the 'prefix' notation, but that choice is immaterial, a 'tree' notation would fit as well; parentheses are mere annotations, for readability).

Now guess that with only five proper names you are able to designate not only those but indeed a countable infinity of objects, among them all the natural numbers. Notice that only known symbols... and *variables* appear above; i.e., you need to choose a slightly wider vocabulary.

2.3.1 Grand-vocabulary

Definition 5 (grand-vocabulary) — *I call every pair, $(\mathcal{V},\mathcal{X})$,*
- *$\mathcal{V}$ being a vocabulary, $\mathcal{V}=(\mathcal{F},a)$,*
- *$\mathcal{X}$ being a finite numbered set of symbols (the variables), $\mathcal{X}=\{\mathrm{x}_l\}_{l\in L}$ ($L\subseteq \mathbf{N}^*$),*

which has this quality:

$$\mathcal{F}\cap\mathcal{X}=\emptyset$$

'a grand-vocabulary'.

Notice that $\mathcal{X}$ is finite: there is not any mathematician, not any programmer, who sees an infinite number of variables in his whole life (I beg that 100 000 variables are enough).

Example — Grand-vocabulary $({}_1\mathcal{V},{}_1\mathcal{X})$

You know the vocabulary ${}_1\mathcal{V}$ (see above). Consider this numbered set of symbols, ${}_1\mathcal{X}$:

u	v	x	y	z
21	22	24	25	26

(L is $\{21,22,24,25,26\}$). The numbering is immaterial, but it matters to choose one, once and for all; I have chosen the lexicographical order.
Verify immediately that the pair $({}_1\mathcal{V},{}_1\mathcal{X})$ is a grand-vocabulary. □

Now examine what symbolic objects you shall qualify as terms.

2.3.2 Symbolic-sequence (notation)

The essential quality of a symbolic sequence is that it is writable: yes, to write it you write a symbol, leave a space, write a symbol etc. (you write-a-symbol m times, $m \in \mathbf{N}^*$).

Example — `joy joy joy` □

Here is a useful notation that I shall apply soon. — Let $\mathcal{O}$ be a set of symbols. Every function

$$\{j \in \mathbf{N} \mid 1 \leq j \leq m\} \to \mathcal{O}$$

m being any natural number, is called '*an $\mathcal{O}$-symbolic-sequence*' (a word). *Its length* is m.

Notation
1 — Let $\mathbf{l}$ be a symbolic sequence; $\lg \mathbf{l}$ is its length.
2 — Let $\mathbf{o}$ be a symbol of $\mathcal{O}$. $\lceil \mathbf{o} \rceil$ is this $\mathcal{O}$-symbolic-sequence:

$$\begin{array}{rcl} \{1\} & \to & \mathcal{O} \\ 1 & \mapsto & \mathbf{o} \end{array}$$

You can generalize easily: $\lceil \mathbf{o}_1\ \mathbf{o}_2 \rceil$, $\lceil \mathbf{o}_1\ \mathbf{o}_2\ \mathbf{o}_3 \rceil$, etc.
3 — Let $\mathbf{l}_1$ be an $\mathcal{O}$-symbolic-sequence; let $\mathbf{l}_2$ be an $\mathcal{O}$-symbolic-sequence. $\lceil \mathbf{l}_1\ \mathbf{l}_2 \rceil$ is this symbolic sequence:

$$\begin{array}{rcl} \{1, 2, \ldots, \lg \mathbf{l}_1 + \lg \mathbf{l}_2\} & \to & \mathcal{O} \\ \{1, 2, \ldots, \lg \mathbf{l}_1\} \ni j & \mapsto & \mathbf{l}_1(j) \\ \{\lg \mathbf{l}_1 + 1, \ldots, \lg \mathbf{l}_1 + \lg \mathbf{l}_2\} \ni j & \mapsto & \mathbf{l}_2(j - \lg \mathbf{l}_1) \end{array}$$

You can generalize easily: $\lceil \mathbf{l}_1\ \mathbf{l}_2\ \mathbf{l}_3 \rceil$, etc.

Example — $\lceil$`joy zer one`$\rceil$ is a ${}_1\mathcal{V}$-symbolic-sequence. □

2.3.3 Term

Among all the $(\mathcal{V} \cup \mathcal{X})$-symbolic-sequences there are the $(\mathcal{V}, \mathcal{X})$-terms.

Definition 6 (term) — *Let $(\mathcal{V}, \mathcal{X})$ be a grand-vocabulary; $\mathcal{V} = (\mathcal{F}, a)$. I call this* 'a $(\mathcal{V}, \mathcal{X})$-term':
- *every symbolic sequence* $\lceil \mathbf{e} \rceil$, $\mathbf{e} \in \mathcal{F}_0$*;*
- *every symbolic sequence* $\lceil \mathbf{x} \rceil$, $\mathbf{x} \in \mathcal{X}$*;*
- *every symbolic sequence* $\lceil \mathbf{f}\ \mathbf{a}_1\ \ldots\ \mathbf{a}_n \rceil$, $\mathbf{f}$ *being an n-ary semi-function-symbol* ($n \in \mathbf{N}^*$), *and* $[\forall i \in \{1, \ldots, n\}$, $\mathbf{a}_i$ *being a* $(\mathcal{V}, \mathcal{X})$*-term*$]$.

Notation — It is convenient to note the interval $\{1, \ldots, n\}$ '*I*', and the symbolic sequence $\lceil \mathbf{a}_1\ \ldots\ \mathbf{a}_n \rceil$ '$\lceil \mathbf{a}_I \rceil$'.

3 INTERPRETING TERMS

At last, you are ready to interpret *every* term *as a semi-function*; i.e., you will make every term, which is a composite of proper names and variables, mean some composite of the semi-functions that you have selected and named at the beginning (the 'primitive' objects).

Let $(\mathcal{V}, \mathcal{X})$ be a grand-vocabulary; $\mathcal{V} = (\mathcal{F}, a)$, $\mathcal{X} = \{\mathrm{x}_l\}_{l \in L}$.

Let $\mathbf{a}$ be a $(\mathcal{V},\mathcal{X})$-term. The set of the variables that appear in $\mathbf{a}$ is a subset of $\mathcal{X}$:

$$\mathrm{Var}\,\mathbf{a} = \{\mathbf{x}_j\}_{j\in J}$$

('J' denotes the set of the numbers of the variables of $\mathbf{a}$; remember that every variable is associated with a distinctive number). I call the cardinal of J 'm': $m = \mathrm{Card}\, J$.

Let $\mathbf{\Omega}$ be a $\mathcal{V}$-magma; $\mathbf{\Omega} = (\Omega, \Phi)$. *Referring to* $\mathbf{\Omega}$*, you are going to interpret the term* $\mathbf{a}$ *as a semi-function* $\Omega^m \rightharpoonup \Omega$, $\tau_{\mathbf{a}}$.
Comment — That notation seems to be reasonable: $\mathbf{a}$ denotes $\tau_{\mathbf{a}}$, whereas every semi-function-symbol $\mathbf{f}$ denotes $\phi_{\mathbf{f}}$. Notice that $\mathbf{a}$ and $\mathbf{f}$ are symbolic objects, whereas $\tau_{\mathbf{a}}$ and $\phi_{\mathbf{f}}$ are semi-functions (a τ is a composite of ϕ).

Examples — 1/0

You know ${}_1\mathcal{V}$, ${}_1\mathcal{X}$ and ${}_1\mathbf{\Omega}$ (see above). The term $\lceil / \ \mathbf{x}\ \mathbf{y}\rceil$ denotes this semi-function:

$$\begin{array}{rcl} \mathbf{R}^2 & \rightharpoonup & \mathbf{R} \\ \mathbf{R}\times\mathbf{R}^* \ni (\xi,\eta) & \mapsto & \xi/\eta \end{array}$$

The term $\lceil / \ \mathbf{one}\ \mathbf{exo}\rceil$ denotes this $\mathbf{R}$-semi-nunction:

$$\begin{array}{rcl} \mathbf{R}^0 & \rightharpoonup & \mathbf{R} \\ \diamond & \mapsto & 1/e \end{array}$$

The term $\lceil / \ \mathbf{one}\ \mathbf{zer}\rceil$ denotes the *empty* $\mathbf{R}$-semi-nunction; i.e., $\tau_{\lceil / \ \mathbf{one}\ \mathbf{zer}\rceil}$ is the semi-function $\mathbf{R}^0 \rightharpoonup \mathbf{R}$ whose domain is empty. □

To interpret the term $\mathbf{a}$, you have to apply a procedure. Here it is. Although avoidable, it is convenient to consider $2+2$ cases.

A — $\mathbf{a}$ is atomic ($\lg \mathbf{a} = 1$)

A-1 — $\mathbf{a}$ is $\lceil \mathbf{e}\rceil$, $\mathbf{e} \in \mathcal{F}_0$

$$\boxed{\lceil \mathbf{e}\rceil \text{ denotes } \phi_{\mathbf{e}} \ (\tau_{\lceil \mathbf{e}\rceil} = \phi_{\mathbf{e}}).}$$

Example — Referring to ${}_1\mathbf{\Omega}$, interpret the term $\lceil \mathbf{exo}\rceil$ as $\tau_{\lceil \mathbf{exo}\rceil} = \phi_{\mathbf{exo}}$, i.e., as the semi-nunction $(\{(\diamond, e)\}, \mathbf{R}^0, \mathbf{R})$. Observe that both the term $\lceil \mathbf{exo}\rceil$ and the symbol '$\mathbf{exo}$' mean a same object. □

A-2 — $\mathbf{a}$ is $\lceil \mathbf{x}\rceil$, $\mathbf{x} \in \mathcal{X}$

$$\boxed{\lceil \mathbf{x}\rceil \text{ denotes } \mathrm{Id}_\Omega \ (\tau_{\lceil \mathbf{x}\rceil} = \mathrm{Id}_\Omega).}$$

Example — Referring to ${}_1\mathbf{\Omega}$, interpret the term $\lceil \mathbf{u}\rceil$ as $\mathrm{Id}_{\mathbf{R}}$ (the $\mathbf{R}$-identity-function). □

B — $\mathbf{a}$ is not atomic; i.e., $\mathbf{a} = \lceil \mathbf{f}\ \mathbf{a}_I\rceil$
Reminder — $\mathbf{f} \in \mathcal{F}_n$, $n \in \mathbf{N}^*$; $I = \{1,\ldots,n\}$; and $[\forall i \in I$, $\mathbf{a}_i$ is a term$]$.

B-1 — $\mathrm{Var}\,\mathbf{a} = \emptyset$ (term without any variable)
Proceed by induction.

Hypothesis — $\forall i \in I$, $\mathbf{a}_i$ denotes $\tau_{\mathbf{a}_i}$, an Ω-semi-nunction.

The term $\lceil \mathbf{f}\ \mathbf{a}_I \rceil$ denotes $\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}$, which is this Ω-semi-nunction: (guess easily the notation, or see below)

if $[\exists i \in I,\ \mathrm{Dom}\,\tau_{\mathbf{a}_i} = \emptyset]$,
then $\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}$ is the empty Ω-semi-nunction, i.e., $\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil} = (\emptyset, \Omega^0, \Omega)$;
else
  if the vector $(\tau_{\mathbf{a}_i}\diamond)_{i\in I} \notin \mathrm{Dom}\,\phi_{\mathbf{f}}$,
  then $\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}$ is the empty Ω-semi-nunction;
  else $\mathrm{Dom}\,\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil} = \Omega^0$ and $\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}\diamond = \phi_{\mathbf{f}}(\tau_{\mathbf{a}_i}\diamond)_{i\in I}$.

Notation — $(\tau_{\mathbf{a}_i}\diamond)_{i\in I} = (\tau_{\mathbf{a}_1}\diamond, \ldots, \tau_{\mathbf{a}_n}\diamond)$.

Examples
1 — Referring to ${}_1\mathbf{\Omega}$, interpret the term `⌈joy one exo⌉` as this *non-empty* $\mathbf{R}$-semi-nunction:

$$\begin{array}{rcl} \mathbf{R}^0 & \to & \mathbf{R} \\ \diamond & \mapsto & \phi_{\mathbf{joy}}(\phi_{\mathbf{one}}\diamond, \phi_{\mathbf{exo}}\diamond) = 1 + \ln e = 2 \end{array}$$

2 — The term `⌈joy one zer⌉` denotes the *empty* $\mathbf{R}$-semi-nunction, for the pair $(1,0)$ is not an element of $\mathrm{Dom}\,\phi_{\mathbf{joy}}$. □

B-2 — $\mathrm{Var}\,\mathbf{a} \neq \emptyset$ (i.e., $\mathbf{a}$ is a term in which at least one variable appears)
Remember that

$$\mathrm{Var}\,\mathbf{a} = \mathrm{Var}\,\lceil \mathbf{f}\ \mathbf{a}_I \rceil = \{\mathbf{x}_j\}_{j\in J}$$

and observe that, in this case, $m = \mathrm{Card}\,J > 0$.
$\forall i \in I$, the set of the variables that appear in $\mathbf{a}_i$ is a subset of $\mathrm{Var}\,\mathbf{a}$:

$$\mathrm{Var}\,\mathbf{a}_i = \{\mathbf{x}_j\}_{j\in J_i}$$

$(J_i \subseteq J)$. I call the cardinal of J_i 'm_i'; observe that $0 \le m_i \le m$ and that not every m_i is 0 (there appears at least some variable somewhere).
Proceed by induction.

Hypothesis — $\forall i \in I$, $\mathbf{a}_i$ denotes $\tau_{\mathbf{a}_i}$, a semi-function $\Omega^{m_i} \to \Omega$.

The term $\lceil \mathbf{f}\ \mathbf{a}_I \rceil$ denotes $\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}$, which is this semi-function $\Omega^m \to \Omega$: (to understand the notation, see below)

(i) $\mathrm{Dom}\,\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil} = \{\,(\xi_j)_{j\in J} \in \Omega^m \mid [\forall i \in I,\ (\xi_j)_{j\in J_i} \in \mathrm{Dom}\,\tau_{\mathbf{a}_i}]$ and $(\ \tau_{\mathbf{a}_i}(\xi_j)_{j\in J_i}\ )_{i\in I} \in \mathrm{Dom}\,\phi_{\mathbf{f}}\}$;

(ii) $\forall (\xi_j)_{j\in J} \in \mathrm{Dom}\,\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}$, $\quad \tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}(\xi_j)_{j\in J} = \phi_{\mathbf{f}}(\ \tau_{\mathbf{a}_i}(\xi_j)_{j\in J_i}\ )_{i\in I}$.

Notation — J is a *non-empty* and finite subset of $\mathbf{N}^*$. I.e., $J = \{j_1, \ldots, j_m\}$, $1 \le j_1 < \ldots < j_m$.
$\forall j \in J$, let ξ_j be an element of Ω.
- $(\xi_j)_{j\in J}$ is the vector $(\xi_{j_1}, \ldots, \xi_{j_m})$ of Ω^m.

Let i be an element of I. Remember that J_i is a subset of J.
- Either $J_i = \emptyset$. — $(\xi_j)_{j\in\emptyset}$ is $\diamond$ $(\in \Omega^0)$.
- Or $J_i = \{j_{i_1}, \ldots, j_{i_q}\}$, $1 \le j_{i_1} < \ldots < j_{i_q}$. — $(\xi_j)_{j\in J_i}$ is the vector $(\xi_{j_{i_1}}, \ldots, \xi_{j_{i_q}})$ of Ω^{m_i}.

Example — The term `⌈joy z ( joy y x )⌉` denotes this semi-function:

$$\begin{array}{rcl} \mathbf{R}^3 & \to & \mathbf{R} \\ D \ni (\xi_{24}, \xi_{25}, \xi_{26}) & \mapsto & \xi_{26} + \ln(\xi_{25} + \ln \xi_{24}) = \phi_{\mathbf{joy}}(\xi_{26}, \phi_{\mathbf{joy}}(\xi_{25}, \xi_{24})) \end{array}$$

(notice that '$\mathtt{x}$' is first, its number being 24, '$\mathtt{y}$' second and '$\mathtt{z}$' third). I.e., this: (the two expressions are synonymous)

$$\begin{array}{rcl} \mathbf{R}^3 & \rightarrow & \mathbf{R} \\ D \ni (\xi,\eta,\zeta) & \mapsto & \zeta + \ln(\eta + \ln\xi) = \phi_{\mathtt{joy}}(\zeta, \phi_{\mathtt{joy}}(\eta,\xi)) \end{array}$$

The domain D is not simple:

$$D = \operatorname{Dom} \tau_{\lceil \mathtt{joy\ z\ joy\ y\ x} \rceil} = \{ (\xi,\eta,\zeta) \in \mathbf{R}^3 \mid \xi \in \mathbf{R}_+^* \text{ and } (\eta + \ln\xi) \in \mathbf{R}_+^* \}$$

(imagine the related volume in the Euclidean space... a region bounded by a vertical cylinder).□

4 BASIC THEOREMS

4.1 The relation between what a term denotes and what a subterm of it denotes

Any symbolic sequence that appears in a term and which itself is a term is called 'a *subterm* of that term'. Is there any relation between what a term denotes and what a subterm of it denotes?

Example

Remember the vocabulary ${}_1\mathcal{V}$ and the ${}_1\mathcal{V}$-magma ${}_1\boldsymbol{\Omega}$. The sequence $\lceil \mathtt{joy\ z\ joy\ y\ x} \rceil$ is a $({}_1\mathcal{V}, {}_1\mathcal{X})$-term, $\lceil \mathtt{joy\ y\ x} \rceil$ a subterm of it. Is there any relation between $\tau_{\lceil \mathtt{joy\ y\ x} \rceil}$ and $\tau_{\lceil \mathtt{joy\ z\ joy\ y\ x} \rceil}$?

Pause for a minute... Then verify easily that the two domains are related in this way:

$$\forall (\xi,\eta,\zeta) \in \operatorname{Dom} \tau_{\lceil \mathtt{joy\ z\ joy\ y\ x} \rceil}, \quad (\xi,\eta) \in \operatorname{Dom} \tau_{\lceil \mathtt{joy\ y\ x} \rceil}$$

(remember that $\operatorname{Dom} \tau_{\lceil \mathtt{joy\ y\ x} \rceil} = \mathbf{R}_+^* \times \mathbf{R}$; and that
$\operatorname{Dom} \tau_{\lceil \mathtt{joy\ z\ joy\ y\ x} \rceil} = \{ (\xi,\eta,\zeta) \in \mathbf{R}^3 \mid \xi \in \mathbf{R}_+^* \text{ and } (\eta + \ln\xi) \in \mathbf{R}_+^* \}$). □

That example is not exceptional. Here is the general statement.

- Let $(\mathcal{V}, \mathcal{X})$ be a grand-vocabulary, $\mathcal{V} = (\mathcal{F}, a)$, $\mathcal{F}_0 \cup \mathcal{X} \neq \emptyset$ (there is at least one term); $\mathcal{X} = \{\mathrm{x}_l\}_{l \in L}$.
- Let $\mathbf{a}$ be a $(\mathcal{V}, \mathcal{X})$-term; $\operatorname{Var} \mathbf{a} = \{\mathrm{x}_j\}_{j \in J}$; $\operatorname{Card} J = m$.
- Let $\mathbf{c}$ be a subterm of $\mathbf{a}$; $\operatorname{Var} \mathbf{c} = \{\mathrm{x}_j\}_{j \in J_s}$. Notice that J_s *(subterm)* is a subset of J.
- Let (Ω, Φ) be a $\mathcal{V}$-magma. Referring to (Ω, Φ), interpret $\mathbf{a}$ as $\tau_{\mathbf{a}}$ and $\mathbf{c}$ as $\tau_{\mathbf{c}}$ (i.e., $\mathbf{a}$ denotes $\tau_{\mathbf{a}}$, and $\mathbf{c}$ $\tau_{\mathbf{c}}$).

Theorem 1

Case 1 — $\operatorname{Var} \mathbf{a} = \emptyset$

If $\diamond \in \operatorname{Dom} \tau_{\mathbf{a}}$, *then* $\diamond \in \operatorname{Dom} \tau_{\mathbf{c}}$.

Case 2 — $\operatorname{Var} \mathbf{a} \neq \emptyset$

$$\forall (\xi_j)_{j \in J} \in \Omega^m, \quad (\xi_j)_{j \in J} \in \operatorname{Dom} \tau_{\mathbf{a}} \quad \Rightarrow \quad (\xi_j)_{j \in J_s} \in \operatorname{Dom} \tau_{\mathbf{c}}$$

(observe that, in this case, $J \neq \emptyset$ and $m \in \mathbf{N}^*$; remember that $\emptyset \subseteq J_s \subseteq J$ and that, by notation, $[\forall (\xi_j)_{j \in J} \in \Omega^m,\ (\xi_j)_{j \in \emptyset} = \diamond]$).

N.B. — I distinguish two cases mostly to make the theorem more readable, *as case 1 is simple and representative* (that comment applies to th. 2 and 3 as well). However, you easily can avoid that distinction by taking advantage of the notation, considering case 2 only and generalizing; compare both halves of this proof to check that over.

Proof

This proof is much simpler than proofs of theorems 2 and 3, yet similar to those as a scale model to the original. It is a good sample to read.

If Dom $\tau_{\mathbf{a}} = \emptyset$, I immediately verify the statement. — If not ($\tau_{\mathbf{a}}$ is not empty), I reason by induction.

Case 1 — I know that $\diamond \in \text{Dom}\ \tau_{\mathbf{a}}$ and want to verify that $\diamond \in \text{Dom}\ \tau_{\mathbf{c}}$.

i — **a** *is atomic*

There is only one subterm of **a**, **a** itself; therefore $\mathbf{c} = \mathbf{a}$, and $\diamond \in \text{Dom}\ \tau_{\mathbf{c}} = \text{Dom}\ \tau_{\mathbf{a}}$.

ii — **a** *is not atomic*

I.e., $\mathbf{a} = \lceil \mathbf{f}\ \mathbf{a}_I \rceil$, $\mathbf{f} \in \mathcal{F}_n$ $(n \in \mathrm{N}^*)$, $I = \{1, \dots, n\}$ and $[\forall i \in I$, $\mathbf{a}_i$ being a $(\mathcal{V}, \mathcal{X})$-term$]$.

Either $\mathbf{c} = \mathbf{a}$ or $\mathbf{c}$ is a subterm of one of the $\mathbf{a}_i$ $(i \in I)$. If $\mathbf{c} = \mathbf{a}$, $\diamond \in \text{Dom}\ \tau_{\mathbf{c}} = \text{Dom}\ \tau_{\mathbf{a}}$. — If $\mathbf{c}$ is a subterm of $\mathbf{a}_{i_0}$ $(i_0 \in I)$, I proceed in this way.

INDUCTION-HYPOTHESIS — $\diamond \in \text{Dom}\ \tau_{\mathbf{a}_{i_0}} \quad \Rightarrow \quad \diamond \in \text{Dom}\ \tau_{\mathbf{c}}$.

$\diamond \in \text{Dom}\ \tau_{\mathbf{a}} = \text{Dom}\ \tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}$; therefore $\diamond \in \text{Dom}\ \tau_{\mathbf{a}_{i_0}}$ (definition of '$\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}$'); therefore $\diamond \in \text{Dom}\ \tau_{\mathbf{c}}$ (induction-hypothesis).

Case 2 — Let $(\xi_j)_{j \in J} \in \text{Dom}\ \tau_{\mathbf{a}}$. I want to verify that $(\xi_j)_{j \in J_s} \in \text{Dom}\ \tau_{\mathbf{c}}$.

i — **a** *is atomic*

There is only one subterm of **a**, **a** itself; therefore $\mathbf{c} = \mathbf{a}$, and $(\xi_j)_{j \in J_s} \in \text{Dom}\ \tau_{\mathbf{c}} = \text{Dom}\ \tau_{\mathbf{a}}$.

ii — **a** *is not atomic*

I.e., $\mathbf{a} = \lceil \mathbf{f}\ \mathbf{a}_I \rceil$, $\mathbf{f} \in \mathcal{F}_n$ $(n \in \mathrm{N}^*)$, $I = \{1, \dots, n\}$ and $[\forall i \in I$, $\mathbf{a}_i$ being a $(\mathcal{V}, \mathcal{X})$-term$]$.

Either $\mathbf{c} = \mathbf{a}$ or $\mathbf{c}$ is a suberm of one of the $\mathbf{a}_i$ $(i \in I)$. If $\mathbf{c} = \mathbf{a}$, $(\xi_j)_{j \in J_s} \in \text{Dom}\ \tau_{\mathbf{c}} = \text{Dom}\ \tau_{\mathbf{a}}$. — If $\mathbf{c}$ is a subterm of $\mathbf{a}_{i_0}$ $(i_0 \in I)$, I proceed in this way.

I call the subset of indices $j \in J$, $\mathbf{x}_j$ appearing in $\mathbf{a}_{i_0}$, 'J_{i_0}'; Var $\mathbf{a}_{i_0} = \{\mathbf{x}_j\}_{j \in J_{i_0}}$. Notice that $J_s \subseteq J_{i_0}$. I call the cardinal of J_{i_0} 'm_{i_0}'.

INDUCTION-HYPOTHESIS — $\forall (\eta_j)_{j \in J_{i_0}} \in \Omega^{m_{i_0}}$,

$$(\eta_j)_{j \in J_{i_0}} \in \text{Dom}\ \tau_{\mathbf{a}_{i_0}} \quad \Rightarrow \quad (\eta_j)_{j \in J_s} \in \text{Dom}\ \tau_{\mathbf{c}}$$

$(\xi_j)_{j \in J} \in \text{Dom}\ \tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}$; therefore $(\xi_j)_{j \in J_{i_0}} \in \text{Dom}\ \tau_{\mathbf{a}_{i_0}}$ (definition of '$\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil}$'); therefore $(\xi_j)_{j \in J_s} \in \text{Dom}\ \tau_{\mathbf{c}}$ (induction-hypothesis). ∎

4.2 The relation between what a term denotes and what a replaceate of it denotes

In most instances you feel free to replace some subterm of a term by an equal term, especially by an equal and simpler term.

Example

You know that $0 + \ln 1 = 0$. Therefore, referring to ${}_1\Omega$ (the real numbers), you find that the terms $\lceil$`joy zer one`$\rceil$ and $\lceil$`zer`$\rceil$ are equal (synonymous).

Now consider ⌈joy (joy zer one) x⌉. Replacing in it ⌈joy zer one⌉ by ⌈zer⌉, you write ⌈joy zer x⌉. Can you verify that $\tau_{\lceil \texttt{joy zer x} \rceil}$ is (the same as) $\tau_{\lceil \texttt{joy joy zer one x} \rceil}$? Yes, you can easily: both terms denote the semi-function ln (logarithm). □

Here is the general statement (see next page).

- Let $(\mathcal{V}, \mathcal{X})$ be a grand-vocabulary, $\mathcal{V} = (\mathcal{F}, a)$, $\mathcal{F}_0 \cup \mathcal{X} \neq \emptyset$ (there is at least one term); $\mathcal{X} = \{\mathbf{x}_l\}_{l \in L}$.
- Let $\mathbf{a}$ be a $(\mathcal{V}, \mathcal{X})$-term; i.e., $\mathbf{a}$ is a symbolic sequence

$$\{1, \ldots, \lg \mathbf{a}\} \to \mathcal{F} \cup \mathcal{X}$$

- Let $\{i_f, \ldots, i_l\}$ be a subinterval of $\{1, \ldots, \lg \mathbf{a}\}$ ($1 \leq i_f \leq i_l \leq \lg \mathbf{a}$; '$f$' stands for 'first', '$l$' for 'last') *such that* the sequence

$$\begin{array}{rcl} \{1, \ldots, i_l - i_f + 1\} & \to & \mathcal{F} \cup \mathcal{X} \\ i & \mapsto & \mathbf{a}(i_f + i - 1) \end{array}$$

  itself is a term. I note it '$\mathbf{c}$'; $\mathbf{c}$ is a subterm of $\mathbf{a}$.
  I say that $\mathbf{c}$ *appears in* $\mathbf{a}$ *at position* i_f. Beware that $\mathbf{c}$ may appear elsewhere as well! I also say that $\mathbf{c}$ *appears in* $\mathbf{a}$ *at area* $\{i_f, \ldots, i_l\}$.
- Let $\mathbf{d}$ be a $(\mathcal{V}, \mathcal{X})$-term.
- Now consider the term that you write in this way: copy the part of $\mathbf{a}$ which is *left of area* $\{i_f, \ldots, i_l\}$, then copy $\mathbf{d}$, then copy the part of $\mathbf{a}$ which is *right of area* $\{i_f, \ldots, i_l\}$ (in $\mathbf{a}$ you have replaced a $\mathbf{c}$ by $\mathbf{d}$); I note that term '$\mathbf{a}'$'.
  Neologisms — I call $\mathbf{a}'$ *'the replaceate'* (think of 'filtrate', 'condensate') and $\mathbf{d}$ *'the replaceand'* (think of 'multiplicand'); $\mathbf{c}$ is the *replaced.*
- I note the subset $\{l \in L \mid \mathbf{x}_l \in \operatorname{Var} \mathbf{c}\}$ (indices of variables that appear in $\mathbf{c}$) 'J_s' *(subterm)*; $\operatorname{Var} \mathbf{c} = \{\mathbf{x}_j\}_{j \in J_s}$.
- I note the subset of indices $l \in L$, the variable $\mathbf{x}_l$ appearing in $\mathbf{a}$ *out of area* $\{i_f, \ldots, i_l\}$, 'J_r' *(rest)*. Verify easily that

$$\operatorname{Var} \mathbf{a} = \{\mathbf{x}_j\}_{j \in J_s \cup J_r}$$

  Notice that, if $\mathbf{c}$ appears twice in $\mathbf{a}$, $J_s \subseteq J_r$.
- I note the subset of indices $\{l \in L \mid \mathbf{x}_l \in \operatorname{Var} \mathbf{d}\}$ 'J_s''; $\operatorname{Var} \mathbf{d} = \{\mathbf{x}_j\}_{j \in J_s'}$. Verify easily that

$$\operatorname{Var} \mathbf{a}' = \{\mathbf{x}_j\}_{j \in J_s' \cup J_r}$$

- I note the cardinal of $J_s \cup J_s' \cup J_r$ 'm_+'.
- Let (Ω, Φ) be a $\mathcal{V}$-magma. Referring to (Ω, Φ), interpret $\mathbf{a}$ as $\tau_{\mathbf{a}}$, $\mathbf{c}$ as $\tau_{\mathbf{c}}$, $\mathbf{d}$ as $\tau_{\mathbf{d}}$, and $\mathbf{a}'$ as $\tau_{\mathbf{a}'}$.

Theorem 2
Case 1 — $m_+ = 0$
If $\diamond \in \operatorname{Dom} \tau_{\mathbf{a}}$ *,*
if $\diamond \in \operatorname{Dom} \tau_{\mathbf{d}}$ *, and*
if $\tau_{\mathbf{c}} \diamond = \tau_{\mathbf{d}} \diamond$ *,*
then $\diamond \in \operatorname{Dom} \tau_{\mathbf{a}'}$ *, and*

$$\tau_{\mathbf{a}'} \diamond = \tau_{\mathbf{a}} \diamond$$

(if the replaced and the replaceand denote a same thing, then the original term and the replaceate denote also a same thing).

Case 2 — $m_+ \in \mathbf{N}^*$
$\forall (\xi_j)_{j \in J_s \cup J'_s \cup J_r} \in \Omega^{m_+}$,
if $(\xi_j)_{j \in J_s \cup J_r} \in \mathrm{Dom}\, \tau_{\mathbf{a}}$,
if $(\xi_j)_{j \in J'_s} \in \mathrm{Dom}\, \tau_{\mathbf{d}}$, *and*
if $\tau_{\mathbf{c}}(\xi_j)_{j \in J_s} = \tau_{\mathbf{d}}(\xi_j)_{j \in J'_s}$,
then $(\xi_j)_{j \in J'_s \cup J_r} \in \mathrm{Dom}\, \tau_{\mathbf{a}'}$, *and*

$$\tau_{\mathbf{a}'}(\xi_j)_{j \in J'_s \cup J_r} = \tau_{\mathbf{a}}(\xi_j)_{j \in J_s \cup J_r}$$

(analogous comment).

N.B. — If $(\xi_j)_{j \in J_s \cup J_r} \in \mathrm{Dom}\, \tau_{\mathbf{a}}$, then $(\xi_j)_{j \in J_s} \in \mathrm{Dom}\, \tau_{\mathbf{c}}$ (theorem 1).

Proof

This proof is long because it is detailed. — I take advantage of the notation and consider both cases simultaneously.

If there is not any tuple $(\xi_j)_{j \in J_s \cup J'_s \cup J_r} \in \Omega^{m_+}$ having this quality:

- $(\xi_j)_{j \in J_s \cup J_r} \in \mathrm{Dom}\, \tau_{\mathbf{a}}$,
- $(\xi_j)_{j \in J'_s} \in \mathrm{Dom}\, \tau_{\mathbf{d}}$, and
- $\tau_{\mathbf{c}}(\xi_j)_{j \in J_s} = \tau_{\mathbf{d}}(\xi_j)_{j \in J'_s}$,

I immediately verify the statement.

If not, let $(\xi_j)_{j \in J_s \cup J'_s \cup J_r}$ be such a tuple. I reason by induction.

I — a *is atomic*

There is only one subterm of $\mathbf{a}$, $\mathbf{a}$ itself. Therefore, $\mathbf{c} = \mathbf{a}$ and $\mathbf{a}' = \mathbf{d}$ $(J_r = \emptyset)$; I immediately verify the statement.

II — a *is not atomic*

I.e., $\mathbf{a} = \lceil \mathbf{f}\ \mathbf{a}_I \rceil$, $\mathbf{f} \in \mathcal{F}_n$ $(n \in \mathbf{N}^*)$, $I = \{1, \ldots, n\}$ and $[\forall i \in I$, $\mathbf{a}_i$ being a $(\mathcal{V}, \mathcal{X})$-term$]$.

Either $\mathbf{c} = \lceil \mathbf{f}\ \mathbf{a}_I \rceil$. Or $\exists i_0 \in I$, area $\{i_f, \ldots, i_l\}$ being included in or equal to area $\{r_{i_0}, r_{i_0} + \lg \mathbf{a}_{i_0} - 1\}$, at which appears $\mathbf{a}_{i_0}$, and $\mathbf{c}$ being a subterm of $\mathbf{a}_{i_0}$ $(r_1 = 2,\ r_2 = \lg \mathbf{a}_1 + 2$, etc.).

If $\mathbf{c} = \lceil \mathbf{f}\ \mathbf{a}_I \rceil$, $\mathbf{a} = \mathbf{c}$ and $\mathbf{a}' = \mathbf{d}$; I immediately verify the statement. — *If not,* I proceed in this way.

I call the replaceate that you write in this way '$\mathbf{a}'_{i_0}$': copy whatever appears at area $\{r_{i_0}, r_{i_0} + \lg \mathbf{a}_{i_0} - 1\}$ *left of area* $\{i_f, \ldots, i_l\}$, then copy $\mathbf{d}$, then copy whatever appears at area $\{r_{i_0}, r_{i_0} + \lg \mathbf{a}_{i_0} - 1\}$ *right of area* $\{i_f, \ldots, i_l\}$.

I call the subset $\{i \in I \mid 1 \leq i < i_0\}$ '$I_\flat$', the subset $\{i \in I \mid i_0 < i \leq n\}$ '$I_\sharp$'. I immediately verify that

$$\mathbf{a} = \lceil \mathbf{f}\ \mathbf{a}_{I_\flat}\ \mathbf{a}_{i_0}\ \mathbf{a}_{I_\sharp} \rceil$$

and that

$$\mathbf{a}' = \lceil \mathbf{f}\ \mathbf{a}_{I_\flat}\ \mathbf{a}'_{i_0}\ \mathbf{a}_{I_\sharp} \rceil$$

$\forall i\, [i \in I,\ i \neq i_0]$, I call the subset of indices $j \in J_r$, $\mathbf{x}_j$ appearing in $\mathbf{a}_i$, 'J_i'. $\mathrm{Var}\, \mathbf{a}_i = \{\mathbf{x}_j\}_{j \in J_i}$. I call the subset of indices $j \in J_r$, $\mathbf{x}_j$ appearing at area $\{r_{i_0}, r_{i_0} + \lg \mathbf{a}_{i_0} - 1\}$ *outside area* $\{i_f, \ldots, i_l\}$, 'J_{r,i_0}'. I verify easily that

$$\mathrm{Var}\, \mathbf{a}_{i_0} = \{\mathbf{x}_j\}_{j \in J_s \cup J_{r,i_0}}$$

that

$$\mathrm{Var}\ \mathbf{a}'_{i_0} = \{\mathbf{x}_j\}_{j\in J'_s\cup J_{r,i_0}}$$

and that

$$J_r = (\bigcup_{i_\flat\in I_\flat} J_{i_\flat}) \cup J_{r,i_0} \cup (\bigcup_{i_\sharp\in I_\sharp} J_{i_\sharp})$$

I call the cardinal of $J_s \cup J'_s \cup J_{r,i_0}$ 'm_{+,i_0}'.

INDUCTION-HYPOTHESIS — $\forall(\eta_j)_{j\in J_s\cup J'_s\cup J_{r,i_0}} \in \Omega^{m_{+,i_0}}$,
if $(\eta_j)_{j\in J_s\cup J_{r,i_0}} \in \mathrm{Dom}\ \tau_{\mathbf{a}_{i_0}}$,
if $(\eta_j)_{j\in J'_s} \in \mathrm{Dom}\ \tau_{\mathbf{d}}$, *and*
if $\tau_{\mathbf{c}}(\eta_j)_{j\in J_s} = \tau_{\mathbf{d}}(\eta_j)_{j\in J'_s}$,
then $(\eta_j)_{j\in J'_s\cup J_{r,i_0}} \in \mathrm{Dom}\ \tau_{\mathbf{a}'_{i_0}}$, *and*

$$\tau_{\mathbf{a}'_{i_0}}(\eta_j)_{j\in J'_s\cup J_{r,i_0}} = \tau_{\mathbf{a}_{i_0}}(\eta_j)_{j\in J_s\cup J_{r,i_0}}$$

$(\xi_j)_{j\in J_s\cup J_r} \in \mathrm{Dom}\ \tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil}$. Therefore,

- $\forall i[i\in I, i\neq i_0]$, $(\xi_j)_{j\in J_i} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i}$;
- $(\xi_j)_{j\in J_s\cup J_{r,i_0}} \in \mathrm{Dom}\ \tau_{\mathbf{a}_{i_0}}$;
- $(\tau_{\mathbf{a}_{i_\flat}}(\xi_j)_{j\in J_{i_\flat}}, \tau_{\mathbf{a}_{i_0}}(\xi_j)_{j\in J_s\cup J_{r,i_0}}, \tau_{\mathbf{a}_{i_\sharp}}(\xi_j)_{j\in J_{i_\sharp}})_{i_\flat\in I_\flat, i_\sharp\in I_\sharp} \in \mathrm{Dom}\ \phi_{\mathbf{f}}$ (guess easily the notation, please);
- $$\begin{aligned} & \tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil}(\xi_j)_{j\in J_s\cup J_r} \\ = \ & \phi_{\mathbf{f}}(\tau_{\mathbf{a}_{i_\flat}}(\xi_j)_{j\in J_{i_\flat}}, \tau_{\mathbf{a}_{i_0}}(\xi_j)_{j\in J_s\cup J_{r,i_0}}, \tau_{\mathbf{a}_{i_\sharp}}(\xi_j)_{j\in J_{i_\sharp}})_{i_\flat\in I_\flat, i_\sharp\in I_\sharp} \end{aligned}$$

(definition of '$\tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil}$').

I infer that

$$(\xi_j)_{j\in J'_s\cup J_{r,i_0}} \in \mathrm{Dom}\ \tau_{\mathbf{a}'_{i_0}}$$

and that

$$\tau_{\mathbf{a}'_{i_0}}(\xi_j)_{j\in J'_s\cup J_{r,i_0}} = \tau_{\mathbf{a}_{i_0}}(\xi_j)_{j\in J_s\cup J_{r,i_0}}$$

(induction-hypothesis).

Now I know this:

- $\forall i[i\in I, i\neq i_0]$, $(\xi_j)_{j\in J_i} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i}$;
- $(\xi_j)_{j\in J'_s\cup J_{r,i_0}} \in \mathrm{Dom}\ \tau_{\mathbf{a}'_{i_0}}$; and
- $$\begin{aligned} & (\tau_{\mathbf{a}_{i_\flat}}(\xi_j)_{j\in J_{i_\flat}}, \tau_{\mathbf{a}'_{i_0}}(\xi_j)_{j\in J'_s\cup J_{r,i_0}}, \tau_{\mathbf{a}_{i_\sharp}}(\xi_j)_{j\in J_{i_\sharp}})_{i_\flat\in I_\flat, i_\sharp\in I_\sharp} \\ = \ & (\tau_{\mathbf{a}_{i_\flat}}(\xi_j)_{j\in J_{i_\flat}}, \tau_{\mathbf{a}_{i_0}}(\xi_j)_{j\in J_s\cup J_{r,i_0}}, \tau_{\mathbf{a}_{i_\sharp}}(\xi_j)_{j\in J_{i_\sharp}})_{i_\flat\in I_\flat, i_\sharp\in I_\sharp} \\ \in \ & \mathrm{Dom}\ \phi_{\mathbf{f}} . \end{aligned}$$

Yes! I have just verified that

$$(\xi_j)_{j\in J'_s\cup J_r} \in \mathrm{Dom}\ \tau_{\mathbf{a}'}$$

(definition of '$\tau_{\mathbf{a}'}$'). Therefore,

$$\begin{aligned} & \tau_{\mathbf{a}'}(\xi_j)_{j\in J'_s\cup J_r} \\ = \ & \phi_{\mathbf{f}}(\tau_{\mathbf{a}_{i_\flat}}(\xi_j)_{j\in J_{i_\flat}}, \tau_{\mathbf{a}'_{i_0}}(\xi_j)_{j\in J'_s\cup J_{r,i_0}}, \tau_{\mathbf{a}_{i_\sharp}}(\xi_j)_{j\in J_{i_\sharp}})_{i_\flat\in I_\flat, i_\sharp\in I_\sharp} \\ = \ & \phi_{\mathbf{f}}(\tau_{\mathbf{a}_{i_\flat}}(\xi_j)_{j\in J_{i_\flat}}, \tau_{\mathbf{a}_{i_0}}(\xi_j)_{j\in J_s\cup J_{r,i_0}}, \tau_{\mathbf{a}_{i_\sharp}}(\xi_j)_{j\in J_{i_\sharp}})_{i_\flat\in I_\flat, i_\sharp\in I_\sharp} \\ = \ & \tau_{\mathbf{a}}(\xi_j)_{j\in J_s\cup J_r} \end{aligned}$$

■

4.3 The relation between what a term denotes and what a substituate of it denotes

Remember that to substitute is not to replace.

Example — To *replace* a ⌈x⌉ (which is a term) by ⌈joy u u⌉ in ⌈joy x x⌉ is to write either ⌈joy (joy u u) x⌉ or ⌈joy x joy u u⌉. To *substitute* ⌈joy u u⌉ for the variable 'x' in ⌈joy x x⌉ is to write ⌈joy joy u u joy u u⌉. □

A term and a substituate of it are obviously linked, the link being the applied substitution. Is there any corresponding relation between the meaning of the term and the meaning of the substituate?

Example

Substituting ⌈joy y x⌉ for the variable 'u', and ⌈z⌉ for 'z', in ⌈joy z u⌉, you write the term ⌈joy z (joy y x)⌉. Then, referring to ${}_1\boldsymbol{\Omega}$ (the real numbers), interpret all those terms.

Remember that

$$\mathrm{Dom}\,\tau_{\lceil \mathtt{joy\ z\ joy\ y\ x}\rceil} = \{\, (\xi_{24},\xi_{25},\xi_{26}) \in \mathbf{R}^3 \mid \xi_{24} \in \mathbf{R}_+^* \text{ and } (\xi_{25} + \ln \xi_{24}) \in \mathbf{R}_+^* \} \qquad (1)$$

(see above) and that

$$\forall (\xi_{24},\xi_{25},\xi_{26}) \in \mathrm{Dom}\,\tau_{\lceil \mathtt{joy\ z\ joy\ y\ x}\rceil},\quad \tau_{\lceil \mathtt{joy\ z\ joy\ y\ x}\rceil}(\xi_{24},\xi_{25},\xi_{26}) = \xi_{26} + \ln(\xi_{25} + \ln \xi_{24}) \quad (2)$$

Now verify this easily (I rewrite formula 1):

$$\begin{array}{lll} \mathrm{Dom}\,\tau_{\lceil \mathtt{joy\ z\ joy\ y\ x}\rceil} = & \{ & (\xi_{24},\xi_{25},\xi_{26}) \in \mathbf{R}^3 \\ & \mid & (\xi_{24},\xi_{25}) \in \mathrm{Dom}\,\tau_{\lceil \mathtt{joy\ y\ x}\rceil} \qquad\qquad \text{(i.e., } \xi_{24} \in \mathbf{R}_+^*\text{)} \\ & & \text{and } \xi_{26} \in \mathrm{Dom}\,\tau_{\lceil \mathtt{z}\rceil} \qquad\qquad \text{(i.e., } \xi_{26} \in \mathbf{R}\text{)} \\ & & \text{and } (\, \tau_{\lceil \mathtt{joy\ y\ x}\rceil}(\xi_{24},\xi_{25}) \,,\, \tau_{\lceil \mathtt{z}\rceil}(\xi_{26}) \,) \in \mathrm{Dom}\,\tau_{\lceil \mathtt{joy\ z\ u}\rceil} \\ & & \qquad\qquad \text{(i.e., } \xi_{25} + \ln \xi_{24} \in \mathbf{R}_+^*\text{)} \\ & \} & \end{array}$$

and this (I rewrite formula 2):

$$\begin{array}{l} \forall (\xi_{24},\xi_{25},\xi_{26}) \in \mathrm{Dom}\,\tau_{\lceil \mathtt{joy\ z\ joy\ y\ x}\rceil}, \\ \quad \tau_{\lceil \mathtt{joy\ z\ joy\ y\ x}\rceil}(\xi_{24},\xi_{25},\xi_{26}) = \tau_{\lceil \mathtt{joy\ z\ u}\rceil}(\, \tau_{\lceil \mathtt{joy\ y\ x}\rceil}(\xi_{24},\xi_{25}) \,,\, \tau_{\lceil \mathtt{z}\rceil}(\xi_{26}) \,) \\ \qquad\qquad\qquad\qquad (\;\; = \xi_{26} + \ln(\xi_{25} + \ln \xi_{24}) \;\;) \end{array}$$

What have you done? You have just verified that what ⌈joy z joy y x⌉, *the substituate,* denotes is connected with what ⌈joy z u⌉, the original term, denotes and to what ⌈joy y x⌉ and ⌈z⌉, *the substitutes,* denote. □

That example is not exceptional. Here is the general statement.

- Let $(\mathcal{V},\mathcal{X})$ be a grand-vocabulary, $\mathcal{X} \neq \emptyset$ (there is at least one variable); $\mathcal{X} = \{\mathrm{x}_l\}_{l\in L}$.
- Let $\mathbf{a}$ be a $(\mathcal{V},\mathcal{X})$-term with at least one variable; $\mathrm{Var}\,\mathbf{a} = \{\mathrm{x}_j\}_{j\in J}$; $J \neq \emptyset$ $(J \subseteq L)$.
- $\forall j \in J$, let $\mathbf{d}_j$ be a $(\mathcal{V},\mathcal{X})$-term (the substitutes). I note the subset $\{l \in L \mid \mathrm{x}_l \in \mathrm{Var}\,\mathbf{d}_j\}$ 'K_j'; $\mathrm{Var}\,\mathbf{d}_j = \{\mathrm{x}_k\}_{k\in K_j}$.
- I note the term that you write by substituting $\mathbf{d}_1,\ldots,\mathbf{d}_j,\ldots$ for $\mathrm{x}_1,\ldots,\mathrm{x}_j,\ldots$ in $\mathbf{a}$ $(j\in J)$ '$\mathbf{a}\{\mathbf{d}_j \circledast \mathrm{x}_j\}_{j\in J}$'.
- I note the union $\bigcup_{j\in J} K_j$ 'K'; I note the cardinal of K 'q'. — Verify this immediately:

$$\mathrm{Var}\,\mathbf{a}\{\mathbf{d}_j \circledast \mathrm{x}_j\}_{j\in J} = \{\mathrm{x}_k\}_{k\in K}$$

- Let (Ω, Φ) be a $\mathcal{V}$-magma. Referring to (Ω, Φ), interpret $\mathbf{a}$ as $\tau_{\mathbf{a}}$, $\mathbf{d}_j$ as $\tau_{\mathbf{d}_j}$; and $\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j \in J}$ as $\tau_{\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j\in J}}$, which is a semi-function $\Omega^q \rightharpoonup \Omega$.

Theorem 3
Case 1 — Var $\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j \in J} = \emptyset$ *(i.e., $\forall j \in J$, $\mathbf{d}_j$ has no variable; $q = 0$)*

If $[\exists j \in J,\ \mathrm{Dom}\, \tau_{\mathbf{d}_j} = \emptyset]$,
then $\mathrm{Dom}\, \tau_{\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j\in J}} = \emptyset$;
else
if $(\ \tau_{\mathbf{d}_j}\diamond\ )_{j \in J} \notin \mathrm{Dom}\, \tau_{\mathbf{a}}$,
then $\mathrm{Dom}\, \tau_{\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j\in J}} = \emptyset$;
else $\diamond \in \mathrm{Dom}\, \tau_{\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j\in J}}$, *and*

$$\tau_{\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j\in J}}\diamond = \tau_{\mathbf{a}}(\ \tau_{\mathbf{d}_j}\diamond\ )_{j \in J}$$

(a substituate denotes a composite of what the original term denotes and of what the substitutes denote).

Case 2 — Var $\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j \in J} \neq \emptyset$ *(observe that, in this case, $q \in \mathbf{N}^*$)*

$$\begin{aligned}\mathrm{Dom}\, \tau_{\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j\in J}} = \{ \ & (\xi_k)_{k\in K} \in \Omega^q \\ | \ & [\forall j \in J,\ (\xi_k)_{k \in K_j} \in \mathrm{Dom}\, \tau_{\mathbf{d}_j}] \\ & \text{and } (\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J} \in \mathrm{Dom}\, \tau_{\mathbf{a}} \\ \} \ & \end{aligned}$$

and

$$\forall (\xi_k)_{k\in K} \in \mathrm{Dom}\, \tau_{\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j\in J}}, \quad \tau_{\mathbf{a}\{\mathbf{d}_j @ \mathbf{x}_j\}_{j\in J}}(\xi_k)_{k\in K} = \tau_{\mathbf{a}}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j \in J}$$

(same comment).

Proof
The proof is long and complex; so do not read it until you have quasi-memorized the notation. See at end (appendix). ∎

5 FINALE

5.1 Application

A famous example

For a long time, logicians have been aware that some formulae seem to be pieces of nonsense, so much so that many of them have avoided to consider that subject altogether. To make it clear, they would show an example, like this famous one:

The king of France who reigns in 1991 is bald.

That sentence *seems* to mean that someone, the king of France who reigns in 1991, has the mentioned property, i.e., is bald. It is equivalent to '$B(k)$', 'B' standing for the predicate is-bald, and 'k' standing for that king.

Now, we all agree that *there is not* any king of France who reigns in 1991 whatsoever. *Does that entail that the formula is a problem and that we must ban it as nonsense? I claim that it is not.* Indeed, you understand easily that sentence now.

- The term 'the king of France who reigns in 1991' is *an empty semi-constant*: it denotes the semi-function $\{\diamond\} \rightharpoonup U$ whose domain is $\emptyset$, U being our universe; i.e., it denotes *the empty U-semi-nunction*.

- *Does that sentence mean a property of our universe ? No.* So let its Boolean value be *no.* Notice that I do not ask whether that sentence is *true* or *false,* which would sound somehow metaphysical, but whether it means a property of some object, the universe of discourse. The answer is ' no '.

To envision the wide scope of that, consider also this ubiquitous term : *' God '.* Depending on whether you believe in God or not, you will judge that ' God ' is either a constant, i.e., there is a God$\diamond$; or an *empty semi-constant,* i.e., Dom $\phi_{\mathrm{God}} = \emptyset$. Tolerant atheists have accepted for a long time to read sentences where ' God ' appears ; they, in their way, understand them.

It is possible to avoid three-valued logic ; it is not necessary to consider $\perp$

I have just replaced the notions *truth* and *falsehood* (true or false) by the notion *applicability* (yes or no), the key question being :

> Does the sentence mean a property of (is applicable to) some universe I am considering ?

Because every term is defined, you will never be led to answer : ' I do not understand the sentence. ' On the contrary, you are always able to answer ' yes ' (true) or ' no ' (false).

> Provided of course that you know the domain of discourse : to the questions ' Is 10 001 prime?' and 'Did ever the king of England speak French ? ', I bet that most German children would answer ' Ich verstehe nicht. '

It is comforting that some authors, for example Burmeister [6], have expressed dislike of or unease with three-valued logic. Although I have not investigated everything, I believe that in every case it is possible to have ' undefined ' equal to ' no '.

However, notice that *both* these sentences ' The goddess Aphrodite loves me. ' and ' The goddess Aphrodite does not love me. ' *do not mean* a property of our universe. So there is a sentence **A**, whose Boolean value is no, the Boolean value of $\neg\mathbf{A}$ being also no (a similar comment applies to the programming logic *PL/CV2* [3]).

G algebra

An area where, for sure, three-valued logic is unnecessary, indeed misleading, is partial algebra (Burmeister [6] and Reichel [22], for example, agree with that). In fact, all what has been presented here is one of the foundations of *G algebra* [20], which is a common superset to both partial algebra [16, 7, 22] and order-sorted algebra [15].

G algebra is a calculus that you can apply to reason formally and uniformly about semi-functions, subsets, and equalities. Every term **a** denotes the object $\tau_{\mathbf{a}}$, which is a composite of the $\phi_{\mathbf{f}}$, the ' primitive ' semi-functions. *A formula where* **a** *appears means a property related to* $\tau_{\mathbf{a}}$*.*

For example (think of the real numbers), $\lceil$`x :: R . y :: R* . $ / y x .`$\rceil$ means that $\mathbf{R} \times \mathbf{R}^* \subseteq \mathrm{Dom}\, \tau_{\lceil \texttt{/ y x} \rceil}$. And $\lceil$`y :: R* . / y y = one .`$\rceil$ means this : (equality called ' existential ' by some people)

(1) $\mathbf{R}^* \subseteq \mathrm{Dom}\, \tau_{\lceil \texttt{/ y y} \rceil}$;

(2) $\forall \xi \in \mathbf{R}^*,\ \xi/\xi = 1$.

Verify immediately that the field of real numbers has indeed those properties. But it has not this one : $\lceil$`x :: R . / x x = one .`$\rceil$.

So, having every term denote an object makes possible that *every* formula be meaningful. You understand every formula, and are able to say *whether* it means a property of some magma you are considering, *or not.*

5.2 An incomplete but significant survey of comparable ideas

5.2.1 Logicians

Although the concept *semi-function* does not belong to the core of mathematical logic (all operations are assumed total), logicians deal all the time with terms and formulae, and their denotations. Therefore it is fundamental to remember mathematical logic and reexamine the different choices.

Church

Alonzo Church [10, p. 21] understands a term as a bunch of functions:

> an n-ary form [i.e., a term where n variables appear] has $n!$ associated n-ary functions, one for each of the permutations of its free variables.

For example, referring to the ring of integers, he would associate to the term ⌈x - y⌉ these two functions:

$$\begin{array}{rcl} \mathbf{Z}^2 & \to & \mathbf{Z} \\ (\xi,\eta) & \mapsto & \xi-\eta \end{array}$$

$$\begin{array}{rcl} \mathbf{Z}^2 & \to & \mathbf{Z} \\ (\xi,\eta) & \mapsto & \eta-\xi \end{array}$$

How is this connected with that? If you order arbitrarily once and for all the whole set of variables, if then among the $n!$ permutations of some subset of variables you choose the ordered one, and choose the associated function as *the* denotation of the term, then you have just executed the very interpretation-procedure that was presented before.

Kleene

Stephen C. Kleene [18] does not state clearly how he interprets terms. He seems even to be hesitant.
1 — Most of the time, he would make the interpretation of a term depend on a choice of objects of the considered universe D (i.e., an assignment of variables). That happens when he wants to know whether a formula is valid: considering some formula $\mathrm{A}(\mathrm{x}_1,\ldots,\mathrm{x}_n)$ and some choice of objects $(x_1,\ldots,x_n)\in D^n$, he would execute an interpretation-procedure to find eventually t or f [18, pp. 168, 389 & 464]. Therefore you are led to think that he does not interpret terms as functions, although he interprets function-symbols as functions.
2 — But in one particular instance, Kleene considers that a term denotes a function [18, p. 198]:

> Under the interpretation of the formal symbolism, a number-theoretic function $\phi(x_1,\ldots,x_n)$ is expressed by a term $\mathrm{t}(\mathrm{x}_1,\ldots,\mathrm{x}_n)$.

Shoenfied

Joseph R. Shoenfield interprets ground terms *only* [23, pp. 18 & 19]. So to interpret a formula, he has to consider all its ground instances ($\mathbf{A}[\mathbf{i}_1,\ldots,\mathbf{i}_n]$ is a ground-instance of some formula $\mathbf{A}[\mathbf{x}_1,\ldots,\mathbf{x}_n]$; p. 19).

Cheng

Jen Huan Cheng [9] shows a superset of predicate logic with equality such that you are able to reason about semi-functions. You consider only universes in which there is ($\perp$) (a conventional object called *'bottom'*) so that you can represent any semi-function by a (total) function in the traditional way. The logic is *three-valued.* As Shoenfield, for example, he only interprets ground terms.

By the way, he makes an informative and detailed survey of the subject, and warns against the confusion between the concepts *function* and *action.*

— A *function* is similar to a set of pairs; the universe you consider is an imaginary set of eternal things (classical, platonistic view).

— An *action* is a transformation performed by a living or a machine; the universe you consider is the one in which you live, or part of it, where everything is changing all the time, including yourself (dynamic, intuitionist view).

Craig

William Craig [12] has proposed an interesting formal calculus to deal with semi-functions and equality, which is similar to G algebra (the main difference is that, applying G algebra, you may consider also subsets of the universe). Every term has a meaning ('1/0' is not nonsense), his interpretation-procedure being similar to mine. However there are a few differences:

— he chooses an infinite denumerable number of variables, $\{\mathbf{x}_i\}_{i\in\mathbf{N}}$;

— he understands an assignment of those variables as an ω-tuple, $(\xi_0, \xi_1, \ldots, \xi_n, \ldots) \in \Omega^{\mathbf{N}}$;

— so that *any* term denotes a semi-function $\Omega^{\mathbf{N}} \rightharpoonup \Omega$.

Notice that, although each semi-function-symbol denotes a finitary semi-function, each term denotes an infinitary, ω-ary, semi-function.

5.2.2 Algebraists

Burris and Sankappanavar

Burris and Sankappanavar do not show one but several denotations of some term, among them you choose according to your considering the term unary, binary, ternary etc. [8, p. 63].

For example, consider this term:

```
+ x y
```

'+' being a binary function symbol, and the set of variables being {'x','y','z','t','u','v'}. Because there are 6 variables, ⌈+ x y⌉ is at the same time binary, ternary, quaternary, etc., and sexanary (but not sexy). Therefore, the term has these five denotations: (refer to the field of real numbers, please)

- ($n = 2$) the function

$$\begin{array}{rcl}\mathbf{R}^2 & \to & \mathbf{R}\\ (\xi,\eta) & \mapsto & \xi+\eta\end{array}$$

- ($n = 3$) the function

$$\begin{array}{rcl}\mathbf{R}^3 & \to & \mathbf{R}\\ (\xi_1,\xi_2,\xi_3) & \mapsto & \xi_1+\xi_2\end{array}$$

- (etc.)
- ($n = 6$) the function

$$\begin{array}{rcl}\mathbf{R}^6 & \to & \mathbf{R}\\ (\xi_1,\xi_2,\ldots,\xi_6) & \mapsto & \xi_1+\xi_2\end{array}$$

Although they do not say it, *the variables are ordered*; i.e., ' x ' is first, ' y ' second etc.

Now you understand easily that any term which is a variable denotes some projection. For example $\lceil \mathbf{x} \rceil$ denotes one of these projections: (choose which one according your considering $\lceil \mathbf{x} \rceil$ unary, binary, etc.)

• ($n = 1$)

$$\begin{array}{ccc} \mathbf{R} & \rightarrow & \mathbf{R} \\ \xi & \mapsto & \xi \end{array}$$

• ($n = 2$)

$$\begin{array}{ccc} \mathbf{R}^2 & \rightarrow & \mathbf{R} \\ (\xi, \eta) & \mapsto & \xi \end{array}$$

• (etc.)

• ($n = 6$)

$$\begin{array}{ccc} \mathbf{R}^6 & \rightarrow & \mathbf{R} \\ (\xi_i)_{i \in \{1,\dots,6\}} & \mapsto & \xi_1 \end{array}$$

Grätzer

George Grätzer [16] adopts the same interpretation-procedure as Burris and S., applying it to partial algebra. A minor difference is terminology: he calls a term which he *chooses* to consider n-ary *'a n-ary polynomial symbol'*. An n-ary polynomial symbol is n-ary, but also $(n+1)$-ary, etc. *Once n is chosen,* an n-ary polynomial symbol denotes an *n-ary polynomial,* i.e., a composite of the projections $\Omega^n \rightarrow \Omega$ and of the fundamental operations.

By the way, he convinces us of the utility of partial algebra: ' [...] the language of partial algebras is the natural one *if we want to talk about subsets of an algebra* [...] even if the subsets are not closed under all operations. '

Cohn

Paul M. Cohn [11] does not specify his interpretation-procedure. But you can understand easily that a term does not mean anything by itself, although a pair *(a term, an assignment of variables)* means some object of the universe (p. 162).

Burmeister

Peter Burmeister [6] does not interpret terms but pairs (a term, an assignment of variables). And the process may fail, i.e., he admits ' the non-existence of the interpretation of a term ' (he would consider ' 1/0 ' as nonsense).

By the way, he expresses that three-valued logic is inappropriate: ' But also the approaches proposed so far by logicians (at least those of which we know) are not satisfactory as far as they immediately use a three-valued logic [...] '.

Later [7], P. Burmeister interprets a term t as $(M : t)^{\mathrm{A}}$, a semi-function by which some assignments of variables are related to some elements of the universe A, the arity of $(M : t)^{\mathrm{A}}$ being the cardinal of M, the set of all the chosen variables (that is very similar to Craig's procedure; see above).

Reichel

What Horst Reichel has presented [22] is close to the subject, and very interesting.

α — His theory is connected with Burmeister's.

β — He chooses an assignment of variables (p. 68) before he interprets a term (p. 73). Not every term can be interpreted, some are nonsense.

γ — The concept *semi-nunction* (nullary semi-function) does not appear.

δ — Reichel considers also *term functions* (def. 2-3-3; p. 79), the shape of which being this:

$$A_v \ni \mathbf{a} \overset{t^A}{\mapsto} t(\tilde{\mathbf{a}}_s) \in A_s$$

a being an assignment of *all* variables (not only of those that appear in the term t), and $t(\tilde{\mathbf{a}}_s)$ being what the pair $(\mathbf{a}, t)$ denotes.

But a term function t^A (his notation) is not a $\tau_{\mathbf{a}}$ (my notation), in spite of a strong likeness. For example, although $\tau_{\mathrm{e}} = \phi_{\mathrm{e}}$ ($\mathrm{e} \in \mathcal{F}_0$) (what the term $\lceil \mathrm{e} \rceil$ denotes is what the constant 'e' denotes), $\sigma^A \neq \sigma^A$ (σ is any function-symbol; 'σ^A' on the left means the term function, whereas 'σ^A' on the right means the fundamental operation).

ϵ — The concept *term function* does not seem to be immediately understandable: H. Reichel considers *equationally partial signature* (ep-signature; def. 2-3-1; p. 78) and *domain conditions*, i.e., equalities, before he considers *term functions*.

ζ — Besides, H. Reichel does not reason too much about term functions. For example, how to interpret the formulae-equalities is not connected with term functions, in fact is explained well before (p. 74).

5.2.3 Other investigators

Broy and *Wirsing* [5] do not state formally how they interpret terms, instead refer to the text of Grätzer [16]. Not every term is defined. Also, they consider a *definedness predicate,*

$$A \models D(t)$$

meaning that the ground term t denotes t^A, an element of some algebra A.

Barringer, Cheng and *Jones* [2] 'accept that certain [terms and certain] formulae do not denote anything', for example $\lceil$`factorial ( -1 )`$\rceil$. Although they do not specify how they interpret terms, you can guess easily how they proceed: they interpret pairs (an assignment of variables, a term). By the way, they show a good reason why you should take an interest in semi-functions: 'Partial functions arise quite naturally with recursive definitions.'

Spivey, explaining *Z,* a specification language [24], discusses about the *partially defined terms* that appear in some equalities you write when specifying a computer program (chapter *Discussion,* section 4-3; pp. 90 and foll.). He comments about the choice of Barringer, Cheng and Jones [2] (see above), then presents the relevant ideas of J.-R. Abrial [1]: a two-valued logic; *validity* instead of *truth*; a partial interpretation-procedure.

Manca, Salibra and *Scollo* [19] avoid partial algebra, avoid considering semi-functions; they do not interpret terms but pairs (an assignment of variables, a term).

Gavilanes-Franco and *Lucio-Carrasco* [14] do consider semi-functions but interpret only pairs (an assignment of variables, a term). In every universe they consider, Ω, there is a conventional object noted '$\perp_\Omega$' (bottom), so that everything is defined; '$1/0$' would denote $\perp_{\mathbf{R}}$. Their logic is three-valued.

Poigné [21] and *Weidenbach* [25] do not interpret terms but pairs (an assignment of variables, a term).

5.3 A few conclusions

5.3.1 Forget about assignments of variables

Many people do not interpret terms but pairs (an assignment of variables, a term). Is that an important difference? should you cherish the concept *assignment of variables*? I do not believe it is or you should, for this reason...

Tuples are enough. For a tuple *is* an assignment of naturals, therefore very similar to an assignment of variables.

Example — The triplet of reals $(0, e, \pi)$ is this function:

$$\begin{array}{rcl} \{1,2,3\} & \to & \mathbf{R}^3 \\ 1 & \mapsto & 0 \\ 2 & \mapsto & e \\ 3 & \mapsto & \pi \end{array}$$

□

You see that a tuple is a sequence. If now you number once and for all the variables you choose, then every tuple will correspond to an assignment of variables, one to one.

Reichel [22] and Craig [12], for example, understand that in a similar way, who choose an infinite denumerable set of variables and consider ω-tuples, i.e., $\mathbf{N}$-sequences, instead of assignments of variables.

5.3.2 Request that interpretation-procedures be explained to you in great detail

An interpretation-procedure is a crucial connecting rod between some calculus (mechanical computations) and your intention (semantics). Computing is transforming forms (terms, formulae, etc.) through text-editing-like operations. So, it has not any value unless you can blindly interpret-believe the final, transformed form you read.

How do you verify that the whole process is correct, trustworthy? By carefully considering and comparing the meanings of the forms you read before and after each step. To do that you have to be absolutely definite about the meanings. Therefore, claiming that some argument is a correctness proof, for a similar reason a completeness proof etc., is a misnomer until you know precisely which interpretation-procedure has been adopted (the strength of a chain is the strength of its weakest link).

Whenever you suspect that there are nonsensical forms, request that the interpretation-procedure be explained to you in great detail, and double-check all the proofs.

5.3.3 Adopt a total interpretation-procedure, and get a bonus

You and I do not like very much partial procedures; they are risky. For that reason many people adopt three-valued logic; that is their way to vote for a total truth-evaluation-procedure: *every* form has a truth value, *yes, no,* or *you-name-it.*

The interpretation-procedure I have shown *is total,* also: *every* form has a meaning, i.e., every term denotes an object. That is my way to vote for a total interpretation-procedure.

Now forget about the details and remember that both procedures are strategic. Is it that you and I request that strategic procedures be total? that procedures meant to be executed by humans, not by machines, be total?

The interesting phenomenon to observe is that, after having adopted a total interpretation-procedure, you find easily a total truth-evaluation-procedure and content yourself with two Boolean values only.

Acknowledgment — The concept *semi-nunction,* which is fundamental, was suggested to me by Jean-Luc Rémy (CRIN-INRIA). I thank Alain Quéré, Pierre Marchand and Joseph Rouyer, who carefully read and commented the first version of this text.

Appendix — Proof of theorem 3

It is not difficult to verify that $\tau_{\mathbf{a}\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J}}$ is some composite of some semi-functions τ. What I found difficult was to indicate its domain, for I had to reason multi-inductively.

I take advantage of the notation and consider both cases simultaneously.

I — $\mathbf{a}$ *is atomic*
I.e $\mathbf{a} = \lceil \mathbf{x} \rceil$, $\mathbf{x} \in \mathcal{X}$.

By definition, $\tau_{\mathbf{a}} = \tau_{\mathbf{x}} = \mathrm{Id}_\Omega$. Let $\mathbf{d}$ be a $(\mathcal{V}, \mathcal{X})$-term; $\mathrm{Var}\,\mathbf{d} = \{\mathbf{x}_k\}_{k\in K}$; $\mathrm{Card}\, K = q$. $\mathbf{a}\{\mathbf{d} \,@\, \mathbf{x}\} = \mathbf{x}\{\mathbf{d} \,@\, \mathbf{x}\} = \mathbf{d}$; $\mathrm{Var}\,\mathbf{a}\{\mathbf{d} \,@\, \mathbf{x}\} = \mathrm{Var}\,\mathbf{d} = \{\mathbf{x}_k\}_K$; $\tau_{\mathbf{a}\{\mathbf{d} \,@\, \mathbf{x}\}} = \tau_{\mathbf{d}}$.

I verify easily that

$$\mathrm{Dom}\,\tau_{\mathbf{a}\{\mathbf{d} \,@\, \mathbf{x}\}} = \mathrm{Dom}\,\tau_{\mathbf{d}} = \left\{ \begin{array}{l} (\xi_k)_K \in \Omega^q \\ \mid \quad (\xi_k)_K \in \mathrm{Dom}\,\tau_{\mathbf{d}} \\ \quad \text{and } \tau_{\mathbf{d}}(\xi_k)_K \in \mathrm{Dom}\,\tau_{\mathbf{a}} = \Omega^0 \end{array} \right\}$$

and that, $\forall (\xi_k)_K \in \mathrm{Dom}\,\tau_{\mathbf{a}\{\mathbf{d} \,@\, \mathbf{x}\}} = \mathrm{Dom}\,\tau_{\mathbf{d}}$,

$$\tau_{\mathbf{a}\{\mathbf{d} \,@\, \mathbf{x}\}}(\xi_k)_K = \tau_{\mathbf{d}}(\xi_k)_K = \mathrm{Id}_\Omega \circ \tau_{\mathbf{d}}(\xi_k)_K = \tau_{\mathbf{a}}(\ \tau_{\mathbf{d}}(\xi_k)_K\ )$$

II — $\mathbf{a}$ *is not atomic*
I.e., $\mathbf{a} = \lceil \mathbf{f}\ \mathbf{a}_I \rceil$, $\mathbf{f} \in \mathcal{F}_n$ $(n \in \mathbf{N}^*)$, $I = \{1, \ldots, n\}$ and $[\forall i \in I$, $\mathbf{a}_i$ being a $(\mathcal{V}, \mathcal{X})$-term$]$.

$\forall i \in I$, I call the subset $\{j \in J \mid \mathbf{x}_j \in \mathrm{Var}\,\mathbf{a}_i\}$ 'J_i'; $\mathrm{Var}\,\mathbf{a}_i = \{\mathbf{x}_j\}_{j\in J_i}$. Then I verify easily that

$$\lceil \mathbf{f}\ \mathbf{a}_I \rceil \{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J = \lceil \mathbf{f}\ \lceil \mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i} \rceil^{i\in I} \rceil$$

(guess easily the notation, please).

I call the union $\bigcup_{j\in J_i} K_j$ '$K(i)$', so that I can write:

$$\mathrm{Var}\,\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i} = \{\mathbf{x}_k\}_{k\in K(i)}$$

I call the cardinal of $K(i)$ '$q(i)$'.

I know that

$$\mathrm{Dom}\,\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J} = \left\{ \begin{array}{l} (\xi_k)_K \in \Omega^q \\ \mid \quad \forall i \in I,\ (\xi_k)_{k\in K(i)} \in \mathrm{Dom}\,\tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i}} \\ \quad \text{and } (\ \tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)}\ )_{i\in I} \in \mathrm{Dom}\,\phi_{\mathbf{f}} \end{array} \right\}; \tag{3}$$

and that

$$\begin{array}{l} \forall (\xi_k)_K \in \mathrm{Dom}\,\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J}, \\ \quad \tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J}(\xi_k)_K = \phi_{\mathbf{f}}(\ \tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)}\ )_{i\in I} \end{array} \tag{4}$$

(definition of '$\tau_{\lceil \mathbf{f}\ \mathbf{a}_I \rceil\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J}$').

I call the subset $\{i \in I \mid \mathrm{Var}\,\mathbf{a}_i = \emptyset\}$ 'I_s' ('s' stands for 'sine', the Latin equivalent to 'without'). Because at least one variable appears in $\lceil \mathbf{f}\ \mathbf{a}_I \rceil$, $I - I_s \neq \emptyset$.

$\forall i \in I_s$, $\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i} = \mathbf{a}_i$ ($\mathbf{a}_i$ has no variable); $J_i = \emptyset$, $K(i) = \emptyset$, $q(i) = 0$; and $\tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i}} = \tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in\emptyset}} = \tau_{\mathbf{a}_i}$.

INDUCTION-HYPOTHESIS — $\forall i \in I - I_s$,

$$\operatorname{Dom} \tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i}} = \{ \ (\xi_k)_{k\in K(i)} \in \Omega^{q(i)} \mid \forall j \in J_i,\ (\xi_k)_{k\in K_j} \in \operatorname{Dom} \tau_{\mathbf{d}_j} \text{ and } (\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J_i} \in \operatorname{Dom} \tau_{\mathbf{a}_i} \ \}$$

and

$$\forall (\xi_k)_{k\in K(i)} \in \operatorname{Dom} \tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i}},$$
$$\tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)} = \tau_{\mathbf{a}_i}^{\mathbf{x}_{J_i}}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J_i}$$

Notice that the induction-hypothesis is connected with substituting in each $\mathbf{a}_i$, $i \in I - I_s$.

II-1 — $\operatorname{Dom} \tau_{[\mathbf{f}\ \mathbf{a}_I]\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J}$

I call the set

$$\{ \ (\xi_k)_K \in \Omega^q \mid [\forall j \in J,\ (\xi_k)_{k\in K_j} \in \operatorname{Dom} \tau_{\mathbf{d}_j}] \text{ and } (\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J} \in \operatorname{Dom} \tau_{[\mathbf{f}\ \mathbf{a}_I]} \ \}$$

'D_1', then want to verify that

$$\operatorname{Dom} \tau_{[\mathbf{f}\ \mathbf{a}_I]\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J} = D_1$$

II-1-1 — $\operatorname{Dom} \tau_{[\mathbf{f}\ \mathbf{a}_I]\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J} \subseteq D_1$?

If $\operatorname{Dom} \tau_{[\mathbf{f}\ \mathbf{a}_I]\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J} = \emptyset$, I have finished. — If not, I proceed in this way. Let $(\xi_k)_K \in \operatorname{Dom} \tau_{[\mathbf{f}\ \mathbf{a}_I]\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_J}$;

$$\forall i \in I,\ (\xi_k)_{k\in K(i)} \in \operatorname{Dom} \tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i}} \tag{5}$$

and

$$(\ \tau_{\mathbf{a}_i\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)}\ )_{i\in I} \in \operatorname{Dom} \phi_{\mathbf{f}} \tag{6}$$

(3).

Let $j_0 \in J$. The variable $\mathbf{x}_{j_0}$ appears in some $\mathbf{a}_{i_0}$, $i_0 \in I - I_s$; therefore $j_0 \in J_{i_0}$; and $K_{j_0} \subseteq K(i_0)$. Therefore

$$(\xi_k)_{k\in K(i_0)} \in \operatorname{Dom} \tau_{\mathbf{a}_{i_0}\{\mathbf{d}_j \,@\, \mathbf{x}_j\}_{j\in J_{i_0}}}$$

(5). Therefore

$$(\xi_k)_{k\in K_{j_0}} \in \operatorname{Dom} \tau_{\mathbf{d}_{j_0}}$$

(induction-hypothesis).

I have just verified that

$$\forall j \in J,\ (\xi_k)_{k\in K_j} \in \operatorname{Dom} \tau_{\mathbf{d}_j}$$

To verify that $(\xi_k)_K \in D_1$, *it remains to* verify that

$$(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J} \in \operatorname{Dom} \tau_{[\mathbf{f}\ \mathbf{a}_I]}$$

i.e.,

$$\forall i \in I,\ (\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J_i} \in \operatorname{Dom} \tau_{\mathbf{a}_i}$$

and

$$(\tau_{\mathbf{a}_i}(\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J_i})_{i\in I} \in \mathrm{Dom}\ \phi_{\mathbf{f}}$$

(definition of '$\tau_{[\mathbf{f}\ \mathbf{a}_I]}$').

- $\forall i \in I_s$ $(J_i = \emptyset, K(i) = \emptyset)$, I know that $\diamond \in \mathrm{Dom}\ \tau_{\mathbf{a}_i}$ (5) ; i.e.,

$$(\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J_i} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i}$$

Besides

$$\tau_{\mathbf{a}_i}(\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J_i} = \tau_{\mathbf{a}_i}\diamond = \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}\diamond = \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)}$$

(trivial reasoning).

- $\forall i \in I - I_s$,

$$(\xi_k)_{k\in K(i)} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}$$

(5) ; therefore

$$(\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J_i} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i}$$

and

$$\tau_{\mathbf{a}_i}(\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J_i} = \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)}$$

(induction-hypothesis).

So (consider both cases simultaneously)

$$\forall i \in I,\ (\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J_i} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i}$$

and

$$(\tau_{\mathbf{a}_i}(\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J_i})_{i\in I} = (\tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)})_{i\in I} \in \mathrm{Dom}\ \phi_{\mathbf{f}} \tag{7}$$

(6).

I have finished to verify that $(\xi_k)_K \in D_1$.

II-1-2 — $D_1 \subseteq \mathrm{Dom}\ \tau_{[\mathbf{f}\ \mathbf{a}_I]\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_J}$?

If $D_1 = \emptyset$, I have finished. — If not, I proceed in this way. Let $(\xi_k)_K \in D_1$.

$$\forall j \in J,\ (\xi_k)_{k\in K_j} \in \mathrm{Dom}\ \tau_{\mathbf{d}_j} \tag{8}$$

therefore no $\tau_{\mathbf{d}_j}$ $(j \in J)$ is empty. Because $(\xi_k)_{k\in K} \in D_1$,

$$(\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J} \in \mathrm{Dom}\ \tau_{[\mathbf{f}\ \mathbf{a}_I]}$$

(definition of 'D_1') ; therefore

$$\forall i \in I,\ (\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J_i} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i} \tag{9}$$

and

$$(\tau_{\mathbf{a}_i}(\tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j})_{j\in J_i})_{i\in I} \in \mathrm{Dom}\ \phi_{\mathbf{f}} \tag{10}$$

(definition of '$\tau_{[\mathbf{f}\ \mathbf{a}_I]}$').

- $\forall i \in I_s$ $(J_i = \emptyset, K(i) = \emptyset)$, I know that $\diamond \in \mathrm{Dom}\ \tau_{\mathbf{a}_i}$ (9) ; i.e.,

$$(\xi_k)_{k\in K(i)} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}$$

Besides

$$\tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)} = \tau_{\mathbf{a}_i}\diamond = \tau_{\mathbf{a}_i}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J_i}$$

(trivial).

• $\forall i \in I - I_s$,

$$\forall j \in J_i \subseteq J,\ (\xi_k)_{k\in K_j} \in \mathrm{Dom}\ \tau_{\mathbf{d}_j}$$

(8) and

$$(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J_i} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i}$$

(9). Therefore

$$(\xi_k)_{k\in K(i)} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}$$

and

$$\tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)} = \tau_{\mathbf{a}_i}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J_i}$$

(induction-hypothesis).

So (consider both cases simultaneously)

$$\forall i \in I,\ (\xi_k)_{k\in K(i)} \in \mathrm{Dom}\ \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}$$

and

$$(\ \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)}\ )_{i\in I} = (\ \tau_{\mathbf{a}_i}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J_i}\ )_{i\in I} \in \mathrm{Dom}\ \phi_{\mathbf{f}}$$

(10).
I have just verified that

$$(\xi_k)_K \in \mathrm{Dom}\ \tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_J}$$

(3).

II-2 — $\forall(\xi_k)_K \in \mathrm{Dom}\ \tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_J}$,

$$\tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_J}(\xi_k)_K = \tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J}\ ?$$

If $\mathrm{Dom}\ \tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_J} = \emptyset$, I have finished. — If not, I proceed in this way. Let $(\xi_k)_K \in \mathrm{Dom}\ \tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_J}$. I know that

$$\tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J} = \phi_{\mathbf{f}}(\ \tau_{\mathbf{a}_i}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J_i}\ )_{i\in I}$$

(definition of '$\tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil}$'). I know that

$$\phi_{\mathbf{f}}(\ \tau_{\mathbf{a}_i}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J_i}\ )_{i\in I} = \phi_{\mathbf{f}}(\ \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)}\ )_{i\in I}$$

(7). I know also that

$$\phi_{\mathbf{f}}(\ \tau_{\mathbf{a}_i\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_{j\in J_i}}(\xi_k)_{k\in K(i)}\ )_{i\in I} = \tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_J}(\xi_k)_K$$

(4). Therefore

$$\tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil}(\ \tau_{\mathbf{d}_j}(\xi_k)_{k\in K_j}\ )_{j\in J} = \tau_{\lceil\mathbf{f}\ \mathbf{a}_I\rceil\{\mathbf{d}_j\ @\ \mathbf{x}_j\}_J}(\xi_k)_K$$

That is it, I have verified everything. Is that joy? ∎

References

N.B. — I have conformed to standard ANSI Z39.29-1977 (American national standard for bibliographic references).

[1] Abrial, J.-R. *The mathematical construction of a program and its application to the construction of mathematics.* Science of computer programming. 4: 45–86; 1984.

[2] Barringer, H.; Cheng, J. H.; Jones, C. B. *A logic covering undefinedness in program proofs.* Acta Informatica. 21: 251–269; 1984. Springer-Verlag.

[3] Constable, R. L.; Johnson, S. D.; Eichenlaub, C. D. *An introduction to the PL/CV2 programming logic.* Springer-Verlag; 1982.

[4] Bourbaki, N. Éléments de mathématique. *Théorie des ensembles.* Paris: Hermann; 1977.

[5] Broy, Manfred; Wirsing, Martin. *Partial abstract types.* Acta informatica. 18: 47–64; 1982. Springer-Verlag.

[6] Burmeister, Peter. *Quasi-equational logic for partial algebras.* Springer-Verlag; 1981; 71–80. (Lecture notes in computer science; 117).

[7] Burmeister, P. *A model-theoretic oriented approach to partial algebras.* Berlin: Akademie-Verlag; 1986. (Mathematical research; 31)

[8] Burris, Stanley; Sankappanavar, H. P. *A course in universal algebra.* Springer-Verlag; 1981. (Graduate texts in mathematics; 78).

[9] Cheng, Jen Huan. *A logic for partial functions.* Manchester, U.K.: Victoria University of Manchester; 1986, January; thesis.

[10] Church, Alonzo. *Introduction to mathematical logic — Vol. 1.* Princeton, New Jersey: Princeton University Press; 1956.

[11] Cohn, Paul M. *Universal algebra.* Revised edition. Dordrecht, Holland: D. Reidel; 1981.

[12] Craig, William. *Near-equational and equational systems of logic of partial functions — 1.* The Journal of symbolic logic. 54(2): 795-827; 1989, September.

[13] Gallier, J. H. *Logic for computer science — Foundations of automatic theorem proving.* New York City: Harper & Row; 1986. (Computer science and technology series; 5).

[14] Gavilanes-Franco, Antonio; Lucio-Carrasco, Francisca. *A first order logic for partial functions.* Theoretical Computer Science. 1990; 74: 37–69.

[15] Goguen, Joseph A.; Meseguer, José. *Order-sorted algebra — I — Partial and overloaded operators, errors and inheritance.* Menlo Park, California: S.R.I. International; 1986, March 25; technical report, draft.

[16] Grätzer, George. *Universal algebra.* D. van Nostrand; 1968. (The university series in higher mathematics).

[17] Halmos, Paul R. *Naïve set theory.* Springer-Verlag; 1974. (Undergraduate texts in mathematics).

[18] Kleene, Stephen Cole. *Introduction to metamathematics.* Amsterdam: North-Holland; 6th reprint; 1971. (Bibliotheca mathematica; I).

[19] Manca, V.; Salibra, A.; Scollo, G. *Equational type logic.* Conference on algebraic methodology and software technology (AMAST); 1989, May; Iowa City, Iowa. Theoretical Computer Science. 1990, December 7; 77(1 & 2).

[20] Mégrelis, Aristide. *Algèbre galactique.* Nancy, France: Université Nancy I; 1990, September; thesis.

[21] Poigné, Axel. *Partial algebras, subsorting, and dependent types — Prerequisites of error handling in algebraic specifications.* Sankt Augustin, Deutschland: G.M.D.; 1989; report (F-2-G-2).

[22] Reichel, Horst. *Initial computability, algebraic specifications, and partial algebras.* Berlin: Akademie-Verlag; 1987. (The international series of monographs on computer science; 2).

[23] Shoenfield, Joseph R. *Mathematical logic.* Reading, Massachusetts: Addison-Wesley; 1967. (Series in logic).

[24] Spivey, J. M. *Understanding Z.* Cambridge University Press; 1988. (Cambridge tracts in theoretical computer science; 3).

[25] Weidenbach, Christoph. *A resolution calculus with dynamic sort structures and partial functions.* Universität Kaiserslautern, Fachbereich Informatik; 1989; SEKI report SR-89-23.

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